# Exploring dynamical stability of one-dimensional non-Hermitian classical systems

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*Abstract* – **This work targets architectured structures, or metamaterials, the underlying couplings of which are created by active external forces. Such metamaterials are usually used to mimic quantum condensed matter systems that interact with the environment, and are mathematically described by non-Hermitian Bloch Hamiltonians. Non-Hermitian systems are being extensively researched both in the quantum and the classical realms, focusing mainly on their topological nature and bulk-boundary-related effects. However, less attention is devoted to their dynamical stability properties, which are crucial for implementation. Here, the stability of a simple, yet fundamental non-Hermitian system, is explored.**

## I. INTRODUCTION

The striking analogy between the electronic bandstructure of solids and the frequency dispersion of classical waves enables mimicking various condensed matter phenomena on intrinsically classical platforms, thus enriching and advancing classical wave control capabilities. The most common topic for the analogy is, probably, the quantum topological wave phenomena [1, 2], resulting in beam-like narrow, robust nonreflecting waves, propagating along interfaces or boundaries, as was realized in acoustic, elastic and electric circuit metamaterials [3–5].

In particular, a considerable effort was devoted to mimicking non-Hermitian quantum systems [6], which are open systems that exchange energy with the environment. These systems are characterized by regions of imaginary spectrum, indicating broken parity-time ($PT$) symmetry phase. The corresponding classical non-Hermitian metamaterials are usually obtained by an underlying interplay of gain and loss at the sites, or by externally induced non-reciprocal couplings between the sites.

Under certain conditions the $PT$-symmetric phase can be restored, featuring a fully real spectrum with isolated exceptional point (EP) singularities, while the system remains non-Hermitian. This special case is of a particular interest, since intricate and extraordinary wave dynamics is unveiled at the vicinity of the EPs, such as unidirectional invisibility and absorption, nonreciprocal transmission, or asymmetric wave propagation [7–13].

One of the most explored attributes of non-Hermitian systems is the topological characteristics of their bandstructure, including bulk-boundary correspondence (or its breakdown), edge states, and boundary-related effects, such as the skin effect [14]. In the analysis of these systems, steady-state time-harmonic solutions are usually assumed without explicit consideration of the underlying dynamical stability, i.e. without verifying the conditions for which the actual response converges to these solutions.

However, dynamical stability is central to non-Hermitian systems, which can be intuitively deduced from the wave amplitude growth in the presence of imaginary spectrum. Here, we consider classical one-dimensional non-Hermitian systems, which are fundamental in non-Hermitian metamaterials research, and constitute the most common platform for the unidirectional and non-reciprocal demonstrations. We study the relation between the occurrence of imaginary frequency dispersion and the poles of the corresponding state-space model, which are direct indicators of dynamical stability.

## II. THE SYSTEM MODEL AND STABILITY DISCUSSION

A non-Hermitian system that is frequently considered in condensed matter research is a one-dimensional $A-B$ dimer lattice of a constant $d$, also dubbed the SSH model [15], subjected to an interlacing gain-loss pattern $\pm i\gamma$.

When the electron hopping has an interlacing strength 1 and $\eta > 1$, this system can be described by the effective Bloch Hamiltonian

$$\widetilde{\mathbf{H}} = \begin{pmatrix} -i\gamma & -\left(\eta + e^{-ikd}\right) \\ -\left(\eta + e^{ikd}\right) & i\gamma \end{pmatrix}. \tag{1}$$

This Hamiltonian is non-Hermitian, since $\widetilde{\mathbf{H}}^\dagger \neq \widetilde{\mathbf{H}}$. The corresponding energy spectrum $E$, which is given by $E = \pm\sqrt{\eta^2 - \gamma^2 + 2\eta\cos(kd) + 1}$, contains complex-valued regions. However, when $\gamma$ and $\eta$ satisfy the special relation $\gamma = \eta - 1$, the spectrum becomes real for all $k$, while the Hamiltonian remains non-Hermitian, and the broken PT-symmetry phase is then restored.

To obtain a classical-mechanical model $\ddot{\mathbf{u}} = D\mathbf{u}$ with frequency dispersion analogous to (1), the complex-valued term $i\gamma$ is usually imitated by a velocity variable $\dot{\mathbf{u}}$. This is due to its Fourier transform $\mathscr{F}\{\dot{u}\} = i\omega u$, which yields the required complex term in steady-state when normalizing by the operating frequency. The resulting $n_{th}$ unit cell dynamics of the classical metamaterial then becomes

$$\begin{cases} \ddot{u}_n^A & = u_{n-1}^B + \eta u_n^B - (1+\eta)u_n^A + \gamma\dot{u}_n^A, \\ \ddot{u}_n^B & = u_{n+1}^A + \eta u_n^A - (1+\eta)u_n^B - \gamma\dot{u}_n^B. \end{cases} \tag{2}$$

For an elastic metamaterial, the nearest neighbor coupling term $\eta$ may indicate a spring of stiffness different than 1. While the loss term $-i\gamma$ represents dissipation, which can be obtained by passive elements, the gain term $+i\gamma$ requires external energy. The dynamics of (2) can be therefore realized using a feedback-based metamaterial [10, 16–18]. By the mapping $E \leftrightarrow \omega^2$, the steady-state spectrum of (2) is similar to that of (1) up to a shift and a square root of the frequency, and replicates the PT-symmetry phase restoration for the relation $\gamma = \eta - 1$. Since in that case the classical spectrum is purely real, the underlying dynamics is expected to be stable.

The dynamical stability is determined by the eigenvalues of the $\mathbf{A}$ matrix (also dubbed by poles) of the overall metamaterial state-space realization $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix}\mathbf{u} & \dot{\mathbf{u}}\end{bmatrix}'$. To test the correlation between the poles and the spectrum, the poles are plotted in Fig. 1 alongside the quantum spectrum of (1), and the classical steady-state spectrum of (2) for $\eta = 2$ and $\gamma = \eta - 1 = 1$.

| Quantum dispersion | Classical dispersion | Dynamical system poles |
|---|---|---|

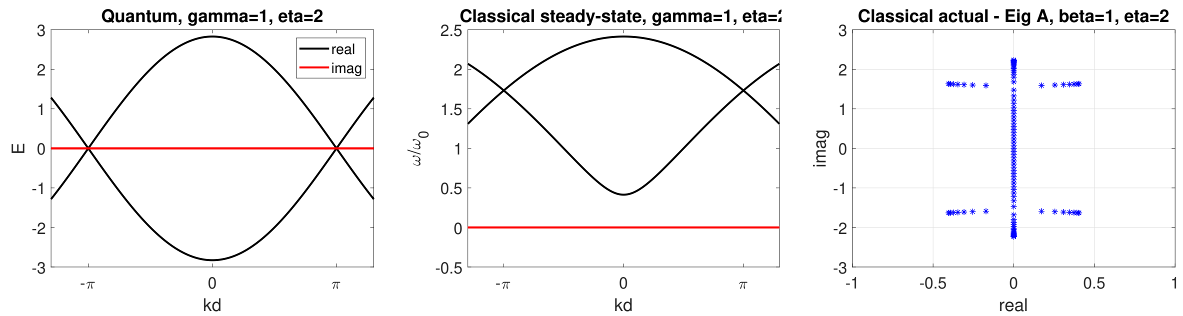


Fig. 1: Left (middle) - dispersion of the quantum (classical) system in the restored PT-symmetric phase. Right - the corresponding state-space poles.

It can be observed from Fig. 1 that the frequency dispersion of the classical model precisely follows the quantum model energy dispersion, in particular at the vicinity of the exceptional points. However, although the spectrum is indeed purely real, as expected, the poles of the dynamical model spread to the right half of the complex plane, indicating instability.

## III. Conclusion

Deploying the quantum PT-symmetry relation directly in the classical metamaterial model does not necessarily guarantee dynamical stability. Therefore, a new relation between the gain parameter $\gamma$ and the nearest neighbor coupling parameter $\eta$ needs to be discovered.

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